

Quantum Geometry of the Dynamical Space-time

P. Leifer¹

Bar-Ilan University, Ramat-Gan, Israel

Abstract

Quantum theory of field (extended) objects without a priori space-time geometry has been represented. Intrinsic coordinates in the tangent fibre bundle over complex projective Hilbert state space $CP(N - 1)$ are used instead of space-time coordinates. The fate of quantum system modeled by the generalized coherent states is rooted in this manifold. Dynamical (state-dependent) space-time arises only at the stage of the quantum “yes/no” measurement. The quantum measurement of the gauge “field shell” of the generalized coherent state is described in terms of the affine parallel transport of the local dynamical variables in $CP(N - 1)$.

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1 Introduction

Strings, membranes are most popular now models of extended quantum objects. They were evoked mostly in order to avoid divergences in quantum field theory. These speculative models give rise for interesting cosmological constructions that are very far from the mature paradigm of our Universe. But I have a doubt that we have now reliable tools for the discussions of global space-time structure (quantum cosmology) since behind achievements in quantum physics we have fundamental unsolved problems in the foundations of quantum mechanics (QM), quantum field theory (QFT), etc.

I would like to show in this work that there are different kind of extended quantum objects which I called “field shell” of generalized coherent states (GCS's) and that in the framework of this model, the local structure of the quantum (or dynamical) space-time has a natural Minkowskian 4D form.

Fast progress achieved in QFT due to efforts of Dirac, Fermi, Jordan, Wigner and other physicists/mathematicians led to jump over gap between one-body problem, which mostly is the academic one, to many-body problem, which is, of course, more practical. However the practical success of the QM and QFT cannot hide the principle difficulties in the one-body quantum problem. I will

¹On leave from Crimea State Engineering and Pedagogical University, Simferopol, Crimea, Ukraine

recall shortly these difficulties: the deep disagreement between both special and general relativity and quantum theory [1], in particular: delocalization of the relativistic wave-function [2]; divergences problem in QFT; the problem of the mass spectrum, etc. The necessity of the quantum geometry as an universal tool evoked to remove difficulties in foundations of quantum theory has been already discussed [3]. One of the main obstacle on this way is just the problem of quantum interactions and their applications to QM measurement problem as the one of the most important in the modern physics [1].

Macroscopic objects are typically well individualizable because of weak interaction and this is the base for our abstraction to material points and further to mathematical points which mutually differ only by space-time coordinates. In quantum area we generally loss this possibility. But of course the conservation of some fundamental dynamical variables does exist. This conservation is related to some internal symmetry of a quantum system and it gives a mechanism of identification of some quantum particle in the framework of one-body problem. One-body measurement (besides bubble chamber measurements) is rarely achievable. The most close to one-body quantum measurement is measurement on coherent beams. I would like to show in present work that the polarization measurements of coherent photons described from the point of view of the breakdown of $U(2)$ symmetry [4], may be generalized to the measurement of arbitrary Hermitian dynamical variable of a “N-level” system as a breakdown of the $G = SU(N)$ symmetry.

2 The Action Quantization

Schrödinger sharply denied the existence of so-called “quantum jumps” during the process of emission/absorption of the quanta of energy (particles) [5, 6]. Leaving the question about the nature of quantum particles outside of consideration, he thought about these processes as a resonance of the de Broglie waves that phenomenologically may look like “jumps” between two “energy levels”. The second quantization method formally avoids these questions but there are at least two reasons for its modification:

First. In the second quantization method one has formally given particles whose properties are defined by some commutation relations between creation-annihilation operators. Note, that the commutation relations are only the simplest consequence of the curvature of the dynamical group manifold in the vicinity of the group’s unit (in algebra). Dynamical processes require, however, finite group transformations and, hence, the global group structure. The main my technical idea is to use vector fields over a group manifold instead of Dirac’s abstract q-numbers. This scheme therefore seeks the dynamical nature of the creation and annihilation processes of quantum particles.

Second. The quantum particles (energy bundles) should gravitate. Hence, strictly speaking, their behavior cannot be described as a linear superposition. Therefore the ordinary second quantization method (creation-annihilation of free particles) is merely a good approximate scheme due to the weakness of

gravity. Thereby the creation and annihilation of particles are time consuming dynamical non-linear processes. So, linear operators of creation and annihilation (in Dirac sense) do exist as approximate quantities.

I discuss here a modification of the “second quantization” procedure. One may image some one-dimension “chain” of the *action states of “elementary quantum motion” (EAS)* $|\hbar a\rangle$ with entire number a of the action quanta. These a, b, c, \dots takes the place of the “principle quantum number” serving as discrete indices $0 \leq a, b, c, \dots < \infty$. Since the action does not create gravity in itself, it is possible to create the linear superposition of $|\hbar a\rangle = (a!)^{-1/2}(\hat{\eta}^+)^a|\hbar 0\rangle$ constituting $SU(\infty)$ multiplete of the Planck’s action quanta operator $\hat{S} = \hbar\hat{\eta}^+\hat{\eta}$ with the spectrum $S_a = \hbar a$ in the separable Hilbert space \mathcal{H} . Therefore, we shall primarily quantize the action, and not the energy. The relative (local) vacuum of some problem is not necessarily the state with minimal energy, it is a state with an extremal of some action functional.

The space-time representation of these states and their coherent superposition $|F\rangle = \sum_{a=0}^{\infty} f^a |\hbar a\rangle$ is postponed on the dynamical stage as it is described below. The superposition principle being applied universally to the EQS $|\hbar a\rangle$, i.e. not only for the description of processes of photon reflection on a mirror, refraction in a glass, etc., [7], but to quantum particles, leads to conclusion that quantum particles arose as some *dynamical processes*. In other words the Schrödinger’s idea of shaping stable wave packet (as the model of material point of the classical oscillator) [8] from oscillators wave functions will applied now to the shaping of *the action amplitudes of an arbitrary quantum system*. Their evolution is defined pure geometrically by some internal unitary gauge “field shell” of the quantum particle. It is close to the old idea of the “field of unitary spin” splitting the hadrons super-multiplete. The stationary processes are represented by stable particles and quasi-stationary processes are represented by unstable resonances.

POSTULATE 1.

I assume that there are elementary quantum states $|\hbar a\rangle, a = 0, 1, \dots$ of an abstract Planck oscillator whose states correspond to the quantum motions with given number of Planck action quanta.

We shall construct non-linear field equations describing energy (frequency) distribution in the “chain” of the EAS’s $|\hbar a\rangle$, whose soliton-like solution provides the quantization of the dynamical variables. Quantum “particles”, and, hence, their numbers arise as some countable solutions of non-linear wave equations. In order to establish acceptable field equations which are capable of intrinsically describing all possible degrees of freedom under intensive interaction we construct a *universal ambient Hilbert state space* \mathcal{H} . We will use *the universality of the action* whose variation is capable of generating any relevant dynamical variable.

Generally the coherent superposition

$$|F\rangle = \sum_{a=0}^{\infty} f^a |\hbar a\rangle, \quad (1)$$

may represent of a ground state or a “vacuum” of some quantum system with the action operator

$$\hat{S} = \hbar A(\hat{\eta}^+ \hat{\eta}). \quad (2)$$

Then one can define the action functional

$$S[|F\rangle] = \frac{\langle F|\hat{S}|F\rangle}{\langle F|F\rangle}, \quad (3)$$

which has the eigen-value $S[|\hbar a\rangle] = \hbar a$ on the eigen-vector $|\hbar a\rangle$ of the operator $\hbar A(\hat{\eta}^+ \hat{\eta}) = \hbar \hat{\eta}^+ \hat{\eta}$ and that deviates in general from this value on superposed states $|F\rangle$ and of course under a different choice of $\hat{S} = \hbar A(\hat{\eta}^+ \hat{\eta}) \neq \hbar \hat{\eta}^+ \hat{\eta}$. In order to study the variation of the action functional on superposed states one need more details on geometry of their superposition.

In fact only finite, say, N elementary quantum states (EQS's) ($|\hbar 0\rangle, |\hbar 1\rangle, \dots, |\hbar(N-1)\rangle$) may be involved in the coherent superposition $|F\rangle$. Then $\mathcal{H} = \mathcal{C}^N$ and the ray space $CP(\infty)$ will be restricted to finite dimensional $CP(N-1)$. Hereafter we will use the indices as follows: $0 \leq a, b \leq N$, and $1 \leq i, k, m, n, s \leq N-1$. This superposition physically corresponds to the complete amplitude of some process of the quantum motion. Sometimes it may be interpreted as a extremal of action functional of some classical variational problem.

3 Non-linear treatment of the eigen-problem

In the famous first report, Schrödinger formulated a conditional variation problem for a “wave function” corresponding to an electron in the Coulomb field. This is the eigen-problem for self-adjoint Hamiltonian in separable Hilbert space [9]. Starting from the internal symmetry given by $SU(N)$ group we will discuss variation problem formulated on finite dimension Hilbert space $\mathcal{H} = \mathcal{C}^N$. In this case the eigen-values of some self-adjoint linear operator \hat{D} are minimal (stationary) values of the quadratic form $\langle F|\hat{D}|F\rangle$ on $CP(N-1)$ [10] achievable at corresponding eigen-vectors of rays. For us will be important the explicit parametrization of these rays by local coordinates in $CP(N-1)$.

Let me start from the eigen-problem for linear operator \hat{D} acting in \mathcal{C}^N . This is the typically linear problem in the framework of the theory of linear operators. Furthermore, physicists have essential simplification of the problem dealing with hermitian or unitary operators. The invariant sub-spaces of these operators are one-dimensional even for degenerated eigen-values, the Jordan cells are one-dimension, and, therefore, these operators are diagonalizable. If so, there is a reasonable question: why one should deep into the non-linear formulation, if the linear one seems to be good enough? It is really so if one deals with the eigen problem for a single matrix of operator. But I seek for the dynamical description of the creation-annihilation processes of particles. This dynamical picture gives a parameterized family of operators; hence arise the instability problem of the

spectrum structure close to the degeneracy and the Berry's anholonomy too [11]. Therefore the non-linear approach is in fact inevitable since the realistic theory in any case requires the dynamical re-definitions of the creation-annihilation operators (the most clear this is realized by the Y_{ni}, Z_{ni} coefficients of Dirac [12]).

The quantum mechanics assumes the priority of the Hamiltonian given by some classical model which henceforth should be "quantized". It is known that this procedure is ambiguous. In order to avoid the ambiguity, I intend to use a *quantum state* itself and the invariant conditions of its conservation and perturbation. These invariant conditions are rooted into the global geometry of the dynamical group manifold. Namely, the geometry of $G = SU(N)$, the isotropy group $H = U(1) \times U(N-1)$ of the pure quantum state, and the coset $G/H = SU(N)/U(1) \times U(N-1)$ geometry, play an essential role in the quantum state evolution (the super-relativity principle [13]). The stationary states (some eigen-states of action operator, i.e. the states of motion with the least action) may be treated as *initial conditions* for GCS evolution. Particularly they may represent a local minimum of energy (vacuum).

Let me assume that $\{|\hbar a \rangle\}_0^{N-1}$ is the basis in Hilbert space \mathcal{H} . Then a typical vector $|F \rangle \in \mathcal{H}$ may be represented as a superposition $|F \rangle = \sum_0^{N-1} f^a |\hbar a \rangle$. The eigen-problem may be formulated for some hermitian dynamical variable \hat{D} on these typical vectors $\hat{D}|F \rangle = \lambda_D |F \rangle$. This equation may be written in components as follows: $\sum_0^{N-1} D_b^a f^b = \lambda_D f^a$, where $D_b^a = \langle a | \hat{D} | b \rangle$.

$$\hat{D} = \sum_{a,b \geq 0} \langle a | \hat{D} | b \rangle \hat{P}_{ab} = \sum_{a,b \geq 0} D_{ab} \hat{P}_{ab} = \sum_{a,b \geq 0} F_D^\alpha \hat{\lambda}_{\alpha,(ab)} \hat{P}_{ab}, \quad (4)$$

where projector is as follows

$$\hat{P}_{ab} = |a \rangle \langle b| = \frac{1}{\sqrt{a!b!}} : (\eta^+)^a \exp(-\eta^+ \eta) (\eta)^b :, \quad (5)$$

and where symbol $: \dots :$ means the normal ordering of operators, and functions F_D^α obey some field equations which will be discussed later. In particular, the Hamiltonian has similar representation

$$\hat{H} = \sum_{a,b \geq 0} \langle a | \hat{H} | b \rangle \hat{P}_{ab} = \sum_{a,b \geq 0} H_{ab} \hat{P}_{ab} = \hbar \sum_{a,b \geq 0} \Omega^\alpha \hat{\lambda}_{\alpha,(ab)} \hat{P}_{ab}, \quad (6)$$

i.e $F_H^\alpha = \hbar \Omega^\alpha$ [14].

One has the spectrum of $\lambda_D : \{\lambda_0, \dots, \lambda_{N-1}\}$ from the equation $\text{Det}(\hat{D} - \lambda_D \hat{E}) = 0$, and then one has the set of equations $\hat{D}|D_p \rangle = \lambda_p |D_p \rangle$, where $p = 0, \dots, N-1$ and $|D_p \rangle = \sum_0^{N-1} g_p^a |\hbar a \rangle$ are eigen-vectors. It is worse while to note here that the solution of this problem gives rather rays than vectors, since eigen-vectors are defined up to the complex factor. In other words we deal with rays or points of the non-linear complex projective space $CP(N-1)$ for

$N \times N$ matrix of the linear operator acting on C^N . The Hilbert spaces of the infinite dimension will be discussed later.

For each eigen-vector $|D_p\rangle$ corresponding λ_p it is possible to chose at least one such component g_p^j of the $|D_p\rangle$, that $|g_p^j| \neq 0$. This choice defines in fact the map $U_{j(p)}$ of the local projective coordinates for each eigen-vecrtor

$$\pi_{j(p)}^i = \begin{cases} \frac{g_p^i}{g_p^j}, & \text{if } 1 \leq i < j \\ \frac{g_p^{i+1}}{g_p^j} & \text{if } j \leq i < N-1 \end{cases} \quad (7)$$

of the ray corresponding $|D_p\rangle$ in $CP(N-1)$. Note, if all $\pi_{j(p)}^i = 0$ it means that one has the “pure” state $|D_p\rangle = g_p^j |j\rangle$ (without summation in j). Any different points of the $CP(N-1)$ corresponds to the GCS’s. They will be treated as self-rays of some deformed action operator. Beside this I will treat the superposition state $|G\rangle = \sum_{a=0}^{N-1} g^a |a\rangle$ as “analytic continuation” of the of eigen-vector for an arbitrary set of the local coordinates. Our aim is to find equations for these operators coinciding with initial operator at original self-ray (initial conditions) given by GCS.

People frequently omit the index p , assuming that $\lambda := \lambda_p$, for $j = 0$. Then they have, say, for the Hamiltonian matrix

$$\hat{H} = \begin{pmatrix} H_{00} & H_{01} & \dots H_{0i} & \dots H_{0N-1} \\ H_{10} & H_{11} & \dots H_{1i} & \dots H_{1N-1} \\ H_{20} & H_{21} & \dots H_{2i} & \dots H_{2N-1} \\ \vdots & \vdots & \vdots & \vdots \\ H_{N-10} & H_{N-11} & \dots H_{N-1i} & \dots H_{N-1N-1} \end{pmatrix} \quad (8)$$

the eigen-problem

$$\begin{aligned} (H_{00} - \lambda)\psi^0 + H_{01}\psi^1 + \dots + H_{0i}\psi^i + \dots + H_{0N-1}\psi^{N-1} &= 0 \\ H_{10}\psi^0 + (H_{11} - \lambda)\psi^1 + \dots + H_{1i}\psi^i + \dots + H_{1N-1}\psi^{N-1} &= 0 \\ H_{20}\psi^0 + H_{21}\psi^1 + (H_{22} - \lambda)\psi^2 + \dots + H_{2i}\psi^i + \dots + H_{2N-1}\psi^{N-1} &= 0 \\ &\vdots \\ &\vdots \\ &\vdots \\ H_{N-10}\psi^0 + H_{N-11}\psi^1 + \dots + H_{N-1i}\psi^i + \dots + (H_{N-1N-1} - \lambda)\psi^{N-1} &= 0, \end{aligned} \quad (9)$$

where $\psi^a := g_0^a$.

In accordance with our assumption the λ is such that $\psi^0 \neq 0$. Let then divide all equations by ψ^0 . Introducing local coordinates $\pi^i = \frac{\psi^i}{\psi^0}$, we get the system of the non-homogeneous equations

$$\begin{aligned} (H_{11} - \lambda)\pi^1 + \dots + H_{1i}\pi^i + \dots + H_{1N-1}\pi^{N-1} &= -H_{10} \\ H_{21}\pi^1 + (H_{22} - \lambda)\pi^2 + \dots + H_{2i}\pi^i + \dots + H_{2N-1}\pi^{N-1} &= -H_{20} \\ &\vdots \end{aligned}$$

$$H_{N-11}\pi^1 + \dots + H_{N-1i}\pi^i + \dots + (H_{N-1N-1} - \lambda)\pi^{N-1} = -H_{N-10}, \quad (10)$$

where the first equation

$$H_{01}\pi^1 + \dots + H_{0i}\pi^i + \dots + H_{0N-1}\pi^{N-1} = -(H_{00} - \lambda) \quad (11)$$

is omitted. If $D = \det(H_{ik} - \lambda\delta_{ik}) \neq 0, i \neq 0, k \neq 0$ then the single defined solutions of this system may be expressed through the Cramer's rule

$$\pi^1 = \frac{D_1}{D}, \dots, \pi^{N-1} = \frac{D_{N-1}}{D}. \quad (12)$$

It is easy to see that these solutions being substituted into the first omitted equation give us simply re-formulated initial characteristic equation of the eigenproblem. Therefore one has the single value nonlinear solution of the eigenproblem instead of the linear one with additional freedom of a complex scale multiplication.

This approach does not give essential advantage for a single operator and it only shows that the formulation in local coordinates is quite natural. But if one tries to understand how the multi-dimensional variation of the hermitian operator included in a parameterized family, the local formulation is inevitable. First of all it is interesting to know the invariants of such variations. In particular the quantum measurement of dynamical variable represented by hermitian $N \times N$ matrix should be described in the spirit of typical polarization measurement of the coherent photons [4]. I will put below the sketch that depicts the operational "travel" of polarization state on the Poincaré sphere.

The initial state $|x\rangle$ is modulated passing through an optically active medium (say using the Faraday effect in YIG film magnetized along the main axes in the z -direction by a harmonic magnetic field with frequency Ω and the angle amplitude β). Formally this process may be described by the action of the unitary matrix \hat{h}_{os_3} belonging to the isotropy group of $|R\rangle$ [13]. Then the coherence vector will oscillate along the equator of the Poincaré sphere. The next step is the dragging of the oscillating state $|x'(t)\rangle = \hat{h}_{os_3}|x\rangle$ with frequency ω up to the "north pole" corresponding to the state $|R\rangle$. In fact this is the motion of the coherence vector. This may be achieved by the variation of the azimuth of the linear polarized state from $\frac{\theta}{2} = -\frac{\pi}{4}$ up to $\frac{\theta}{2} = \frac{\pi}{4}$ with help of the dense flint of appropriate length embedded into the sweeping magnetic field. Further this beam should pass the $\lambda/4$ plate. This process of variation of the ellipticity of the polarization ellipse may be described by the unitary matrix $\hat{b}_{os'_1}$ belonging to the coset homogeneous sub-manifold $U(2)/U(1) \times U(1) = CP(1)$ of the dynamical group $U(2)$ [13]. This dragging without modulation leads to the evolution of the initial state along the geodesic of $CP(1)$ and the trace of the coherent vector is the meridian of the Poincaré sphere between the equator and one of the poles. The modulation deforms both the geodesic and the corresponding trace of the coherence vector on the Poincaré sphere during such unitary evolution.

The action of the $\lambda/4$ plate depends upon the state of the incoming beam (the relative orientation of the fast axes of the plate and the polarization of the beam). Furthermore, only relative phases and amplitudes of photons in the beam have a physical meaning for the $\lambda/4$ plate. Neither the absolute amplitude (intensity of the beam), nor the general phase affect the polarization character of the outgoing state. It means that the device action depends only upon the local coordinates $\pi^1 = \frac{\Psi^1}{\Psi^0} \in CP(1)$. Small relative re-orientation of the $\lambda/4$ plate and the incoming beam leads to a small variation of the outgoing state. This means that the $\lambda/4$ plate re-orientation generates the tangent vector to $CP(1)$. It is natural to discuss the two components of such a vector: velocities of the variations of the ellipticity and of the azimuth (inclination) angle of the polarization ellipse. They are examples of LDV. The comparison of such dynamical variables for different coherent states requires that affine parallel transport agrees with the Fubini-Study metric.

As far as I know the generalized problem of the quantum measurement of an arbitrary hermitian dynamical variable $\hat{H} = E^\alpha \hat{\lambda}_\alpha$, $\hat{\lambda}_\alpha \in AlgSU(N)$ in the operational manner given above was never done. It is solved here by the exact analytical diagonalization of an hermitian matrix. Previously this problem was solved partly in the works [15, 16, 17]. Geometrically it looks like embedding “the ellipsoid of polarizations” into the iso-space of the adjoint representation of $SU(N)$. This ellipsoid is associated with the quadric form $\langle F | \hat{H} | F \rangle = \sum_1^{N^2-1} E^\alpha \langle F | \hat{\lambda}_\alpha | F \rangle = H_{ab}(E^\alpha) f^{a*} f^b$ depending on $N^2 - 1$ real parameters E^α . The shape of this ellipsoid with N main axes is giving by the $2(N - 1)$ parameters of the coset transformations $G/H = SU(N)/S[U(1) \times U(N - 1)] = CP(N - 1)$ relate to the $(N - 1)$ complex local coordinates of the eigen-state of \hat{H} in $CP(N - 1)$. Its orientation in iso-space R^{N^2-1} is much more complicated than it was in the case of R^3 . It is given by generators of the isotropy group containing $N - 1 = rank(AlgSU(N))$ independent parameters of “rotations” about commutative operators $\hat{\lambda}_3, \hat{\lambda}_8, \hat{\lambda}_{15}, \dots$ and $(N - 1)(N - 2)$ parameters of rotations about non-commutative operators. All these $(N - 1)^2 = (N - 1) + (N - 1)(N - 2)$ gauge angles of the isotropy group $H = S[U(1) \times U(N - 1)]$ of the eigen-state giving orientation of this ellipsoid in iso-space R^{N^2-1} will be calculated now during the process of analytical diagonalization of the hermitian matrix $H_{ab} = \langle a | \hat{H} | b \rangle$ corresponding to some dynamical variable \hat{H} .

Stage 1. Reduction of the general Hermitian Matrix to three-diagonal form.

Let me start from general hermitian $N \times N$ matrix

$$\hat{H} = \begin{pmatrix} H_{00} & H_{01} & \dots H_{0i} & \dots H_{0N-1} \\ H_{10} & H_{11} & \dots H_{1i} & \dots H_{1N-1} \\ H_{20} & H_{21} & \dots H_{2i} & \dots H_{2N-1} \\ \vdots & \vdots & \vdots & \vdots \\ H_{N-10} & H_{N-11} & \dots H_{N-1i} & \dots H_{N-1N-1} \end{pmatrix}. \quad (13)$$

One should choose some basis in C^N . I will take the standard basis

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \dots, |N\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}. \quad (14)$$

Now if one choose, say, $N = 3$ then the standard Gell-Mann $\hat{\lambda}$ matrices may be distinguished into the two sets in respect with for example the state $|1\rangle$: B-set $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_4, \hat{\lambda}_5$ whose exponents act effectively on the $|1\rangle$, and the H-set $\hat{\lambda}_3, \hat{\lambda}_8, \hat{\lambda}_6, \hat{\lambda}_7$, whose exponents that leave $|1\rangle$ intact. For any finite dimension N one may define the “I-spin” ($1 \leq I \leq N$) as an analog of the well known “T-, U-, V- spins” of the $SU(3)$ theory using the invariant character of the commutation relations of B- and H-sets

$$[B, B] \in H, \quad [H, H] \in H, \quad [B, H] \in B. \quad (15)$$

Let me now to represent our hermitian matrix in following form

$$\hat{H} = \begin{pmatrix} 0 & H_{01} & \dots H_{0i} & \dots H_{0N-1} \\ H_{10} & 0 & \dots 0 & \dots 0 \\ H_{20} & 0 & \dots 0 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ H_{N-10} & 0 & \dots 0 & \dots 0 \end{pmatrix}_B + \begin{pmatrix} H_{00} & 0 & \dots 0 & \dots 0 \\ 0 & H_{11} & \dots H_{1i} & \dots H_{1N-1} \\ 0 & H_{21} & \dots H_{2i} & \dots H_{2N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & H_{N-11} & \dots H_{N-1i} & \dots H_{N-1N-1} \end{pmatrix}_H. \quad (16)$$

In respect with ket $|1\rangle$ one may to classify the first matrix as B -type and the second one as a matrix of the H -type. I will apply now the “squeezing ansatz” [13, 17]. The first “squeezing” unitary matrix is

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 1 & 0 & 0 \\ . & . & . & . & 0 & \cos \phi_1 & e^{i\psi_1} \sin \phi_1 \\ 0 & 0 & . & . & 0 & -e^{-i\psi_1} \sin \phi_1 & \cos \phi_1 \end{pmatrix}. \quad (17)$$

The transformation of similarity being applied to our matrix gives $\hat{H}_1 = \hat{U}_1^\dagger \hat{H} \hat{U}_1$ with the result for \hat{H}_B shown for simplicity in the case $N = 4$

$$\hat{H}_{B1} = \begin{pmatrix} 0 & H_{01} & \tilde{H}_{02} & \tilde{H}_{03} \\ H_{01}^* & 0 & 0 & 0 \\ \tilde{H}_{02}^* & 0 & 0 & 0 \\ \tilde{H}_{03}^* & 0 & 0 & 0 \end{pmatrix}, \quad (18)$$

where $\tilde{H}_{02} = H_{02} \cos \phi - H_{03} \sin \phi e^{-i\psi}$ and $\tilde{H}_{03} = H_{02} \sin \phi e^{i\psi} + H_{03} \cos \phi$. Now one has solve two “equations of annihilation” of $\Re(H_{02} \sin(\phi) e^{i\psi} + H_{03} \cos(\phi)) = 0$ and $\Im(H_{02} \sin(\phi) e^{i\psi} + H_{03} \cos(\phi)) = 0$ in order to eliminate the last element of the first row and its hermitian conjugate [17]. This gives us ϕ'_1 and ψ'_1 . I will put $H_{02} = \alpha_{02} + i\beta_{02}$ and $H_{03} = \alpha_{03} + i\beta_{03}$, then the solution of the “equations of annihilation” is as follows:

$$\begin{aligned} \phi'_1 &= \arctan \sqrt{\frac{\alpha_{03}^2 + \beta_{03}^2}{\alpha_{02}^2 + \beta_{02}^2}}, \\ \psi'_1 &= \arctan \frac{\alpha_{03}\beta_{02} - \alpha_{02}\beta_{03}}{\sqrt{(\alpha_{02}^2 + \beta_{02}^2)(\alpha_{03}^2 + \beta_{03}^2)}}. \end{aligned} \quad (19)$$

This transformation acts of course on the second matrix \hat{H}_H , but it easy to see that its structure is intact. The next step is the similarity transformations given by the matrix with the diagonally shifted transformation block

$$\hat{U}_2 = \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 1 & 0 & 0 \\ . & . & . & . & 0 & \cos \phi_2 & e^{i\psi_2} \sin \phi_2 \\ 0 & 0 & . & . & 0 & -e^{-i\psi_2} \sin \phi_2 & \cos \phi_2 \\ 0 & . & . & . & 0 & 0 & 1 \end{pmatrix} \quad (20)$$

and the similar evaluation of ψ'_2, ϕ'_2 . Generally one should make $N - 2$ steps in order to annulate $N - 2$ elements of the first row. The next step is to represent

our transformed $\hat{H}_1 = \hat{U}_1^\dagger \hat{H} \hat{U}_1$ as follows:

$$\hat{H}_1 = \begin{pmatrix} 0 & \tilde{H}_{01} & 0 & \dots 0 \\ \tilde{H}_{10} & 0 & \tilde{H}_{12} & \dots \tilde{H}_{1N-1} \\ 0 & \tilde{H}_{21} & 0 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \tilde{H}_{N-1,1} & \dots 0 & \dots 0 \end{pmatrix}_B + \begin{pmatrix} H_{00} & 0 & \dots 0 & \dots 0 \\ 0 & \tilde{H}_{11} & \dots 0 & \dots 0 \\ 0 & 0 & \dots \tilde{H}_{2i} & \dots \tilde{H}_{2N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots \tilde{H}_{N-1,i} & \dots \tilde{H}_{N-1,N-1} \end{pmatrix}_H. \quad (21)$$

Now one should applied the squeezing ansatz in $N - 3$ steps for second row, etc., generally one has $(N - 1)(N - 2)$ orientation angles. Thereby we come to the three-diagonal form of the our matrix.

Stage 2. Diagonalization of the three-diagonal form.

The eigen-problem for the three-diagonal hermitian matrix is well know, but I will put it here for the completeness. The eigen- problem $(\hat{H} - \lambda \hat{E})|\xi\rangle = 0$ for the three-diagonal matrix has the following form

$$\begin{pmatrix} \tilde{H}_{00}\xi^0 & \tilde{H}_{01}\xi^1 & 0 & \cdot & \cdot & \cdot & 0 \\ \tilde{H}_{01}^*\xi^0 & \tilde{H}_{11}\xi^1 & \tilde{H}_{12}\xi^2 & 0 & \cdot & \cdot & 0 \\ 0 & \tilde{H}_{12}^*\xi^1 & \tilde{H}_{22}\xi^2 & \tilde{H}_{23}\xi^3 & \cdot & \cdot & 0 \\ 0 & 0 & \tilde{H}_{23}^*\xi^2 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \tilde{H}_{N-1,N-2}\xi^{N-1} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \tilde{H}_{N-1,N-1}\xi^{N-1} \end{pmatrix} = \begin{pmatrix} \lambda\xi^0 \\ \lambda\xi^1 \\ \lambda\xi^2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda\xi^{N-1} \end{pmatrix}. \quad (22)$$

Since $\xi^1 = \frac{\lambda - \tilde{H}_{00}}{\tilde{H}_{01}}\xi^0$, etc., one has the recurrent relations between all components of the eigen-vector corresponding to given λ . Thereby only $N - 1$ complex local coordinates $(\pi^1 = \frac{\xi^1}{\xi^0}, \dots, \pi^{N-1} = \frac{\xi^{N-1}}{\xi^0})$ giving the shape of the ellipsoid of polarization have invariant sense as it was mentioned above.

Stage 3. The coset “force” acting during a measurement

The real measurement assumes some interaction of the measurement device and incoming state. If we assume for simplicity that incoming state is $|1\rangle$ (modulation, etc. are neglected), then all transformations from H -subalgebra will leave it intact. Only the coset unitary transformations

$$\hat{T}(t, g) = \begin{pmatrix} \cos gt & \frac{-p^{1*}}{g} \sin gt & \frac{-p^{2*}}{g} \sin gt & \cdot & \frac{-p^{N-1*}}{g} \sin gt \\ \frac{p^1}{g} \sin gt & 1 + [\frac{|p^1|}{g}]^2 (\cos gt - 1) & [\frac{p^1 p^{2*}}{g}]^2 (\cos gt - 1) & \cdot & [\frac{p^1 p^{N-1*}}{g}]^2 (\cos gt - 1) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{p^{N-1}}{g} \sin gt & [\frac{p^{1*} p^{N-1}}{g}]^2 (\cos gt - 1) & \cdot & \cdot & 1 + [\frac{|p^{N-1}|}{g}]^2 (\cos gt - 1) \end{pmatrix} \quad (23)$$

where $g = \sqrt{|p^1|^2 + \dots + |p^{N-1}|^2}$ will effectively to variate this state dragging it along one of the geodesic in $CP(N-1)$ [13]. This matrix describe the process of the transition from one pure state to another, in particular between two eigen-states connected by the geodesic. This means that these transformations deform the ellipsoid. All possible shapes of these ellipsoids are distributed along a single geodesic.

Generally, in the dynamical situation this “stationary” global procedure is not applicable and one should go to the local analog of λ -matrices, i.e. $SU(N)$ generators and related dynamical variables should be parameterized by the local quantum states coordinates $(\pi^1, \dots, \pi^{N-1})$.

4 Local dynamical variables

The state space \mathcal{H} with finite action quanta is a stationary construction. We introduce dynamics *by the velocities of the GCS variation* representing some “elementary excitations” (quantum particles). Their dynamics is specified by the Hamiltonian, giving time variation velocities of the action quantum numbers in different directions of the tangent Hilbert space $T_{(\pi^1, \dots, \pi^{N-1})} CP(N-1)$ which takes the place of the ordinary linear quantum state space as will be explained below. The rate of the action variation gives the energy of the “particles” whose expression should be established by some wave equations.

The local dynamical variables correspond to the internal $SU(N)$ group of the GCS and their breakdown should be expressed now in terms of the local coordinates π^k . The Fubini-Study metric

$$G_{ik^*} = [(1 + \sum |\pi^s|^2) \delta_{ik} - \pi^{i*} \pi^k] (1 + \sum |\pi^s|^2)^{-2} \quad (24)$$

and the affine connection

$$\Gamma_{mn}^i = \frac{1}{2} G^{ip^*} \left(\frac{\partial G_{mp^*}}{\partial \pi^n} + \frac{\partial G_{p^*n}}{\partial \pi^m} \right) = - \frac{\delta_m^i \pi^{n*} + \delta_n^i \pi^{m*}}{1 + \sum |\pi^s|^2} \quad (25)$$

in these coordinates will be used. Hence the internal dynamical variables and their norms should be state-dependent, i.e. local in the state space [4, 13]. These local dynamical variables realize a non-linear representation of the unitary global $SU(N)$ group in the Hilbert state space C^N . Namely, $N^2 - 1$ generators of $G = SU(N)$ may be divided in accordance with the Cartan decomposition:

$[B, B] \in H, [B, H] \in B, [B, B] \in H$. The $(N-1)^2$ generators

$$\Phi_h^i \frac{\partial}{\partial \pi^i} + c.c. \in H, \quad 1 \leq h \leq (N-1)^2 \quad (26)$$

of the isotropy group $H = U(1) \times U(N-1)$ of the ray (Cartan sub-algebra) and $2(N-1)$ generators

$$\Phi_b^i \frac{\partial}{\partial \pi^i} + c.c. \in B, \quad 1 \leq b \leq 2(N-1) \quad (27)$$

are the coset $G/H = SU(N)/S[U(1) \times U(N-1)]$ generators realizing the breakdown of the $G = SU(N)$ symmetry of the GCS. Furthermore, the $(N-1)^2$ generators of the Cartan sub-algebra may be divided into the two sets of operators: $1 \leq c \leq N-1$ ($N-1$ is the rank of $AlgSU(N)$) Abelian operators, and $1 \leq q \leq (N-1)(N-2)$ non-Abelian operators corresponding to the non-commutative part of the Cartan sub-algebra of the isotropy (gauge) group. Here Φ_σ^i , $1 \leq \sigma \leq N^2-1$ are the coefficient functions of the generators of the non-linear $SU(N)$ realization. They give the infinitesimal shift of the i -component of the coherent state driven by the σ -component of the unitary multipole field Ω^α rotating the generators of $AlgSU(N)$ and they are defined as follows:

$$\Phi_\sigma^i = \lim_{\epsilon \rightarrow 0} \epsilon^{-1} \left\{ \frac{[\exp(i\epsilon\lambda_\sigma)]_m^i g^m}{[\exp(i\epsilon\lambda_\sigma)]_m^j g^m} - \frac{g^i}{g^j} \right\} = \lim_{\epsilon \rightarrow 0} \epsilon^{-1} \{ \pi^i(\epsilon\lambda_\sigma) - \pi^i \}, \quad (28)$$

[13]. Then the sum of the N^2-1 the energies associated with intensity of deformations of the GCS is represented by the local Hamiltonian vector field \vec{H} which is linear in the partial derivatives $\frac{\partial}{\partial \pi^i} = \frac{1}{2}(\frac{\partial}{\partial \Re \pi^i} - i\frac{\partial}{\partial \Im \pi^i})$ and $\frac{\partial}{\partial \pi^{*i}} = \frac{1}{2}(\frac{\partial}{\partial \Re \pi^i} + i\frac{\partial}{\partial \Im \pi^i})$. In other words it is the tangent vector to $CP(N-1)$

$$\vec{H} = T_c + T_q + V_b = \hbar\Omega^c \Phi_c^i \frac{\partial}{\partial \pi^i} + \hbar\Omega^q \Phi_q^i \frac{\partial}{\partial \pi^i} + \hbar\Omega^b \Phi_b^i \frac{\partial}{\partial \pi^i} + c.c. \quad (29)$$

The characteristic equations for the PDE $\vec{H}|E\rangle = E|E\rangle$ give the parametric representations of their solutions in $CP(N-1)$. We will identify the parameter τ in these equations with a “universal time of evolution” such as the world time [18]. This time is the measure of the GCS variation, i.e. it is a measure of the distance in $CP(N-1)$ (the length of the evolution trajectory in the Fubini-Study metric) expressed in time units. The energy quantization will be discussed elsewhere.

In order to express some eigen-vector in the local coordinates, I put

$$|D_p(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1})\rangle = \sum_0^{N-1} g^a(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1}) |a\rangle, \quad (30)$$

where $\sum_{a=0}^{N-1} |g^a|^2 = R^2$, and

$$g^0(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1}) = \frac{R^2}{\sqrt{R^2 + \sum_{s=1}^{N-1} |\pi_{j(p)}^s|^2}}. \quad (31)$$

For $1 \leq i \leq N-1$ one has

$$g^i(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1}) = \frac{R\pi_{j(p)}^i}{\sqrt{R^2 + \sum_{s=1}^{N-1} |\pi_{j(p)}^s|^2}}, \quad (32)$$

i.e. $CP(N-1)$ is embedded in the Hilbert space $\mathcal{H} = C^N$. Hereafter I will suppose $R = 1$.

Now we see that all eigen-vectors corresponding to different eigen-values (even under the degeneration) are applied to different points $(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1})$ of the $CP(N-1)$. Nevertheless the eigen-vectors $|D_p(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1})\rangle$ are mutually orthogonal in $\mathcal{H} = C^N$ if \hat{H} is hermitian Hamiltonian. Therefore one has the “splitting” or delocalization of degenerated eigen-states in $CP(N-1)$. Thus the local coordinates π^i gives the convenient parametrization of the $SU(N)$ action as one will see below.

Let me assume that $|G\rangle = \sum_{a=0}^{N-1} g^a |a\hbar\rangle$ is a “ground state” of some the least action problem. Then the velocity of the ground state evolution relative world time is given by the formula

$$|H\rangle = \frac{d|G\rangle}{d\tau} = \frac{\partial g^a}{\partial \pi^i} \frac{d\pi^i}{d\tau} |a\hbar\rangle = |T_i\rangle \frac{d\pi^i}{d\tau} = H^i |T_i\rangle, \quad (33)$$

is the tangent vector to the evolution curve $\pi^i = \pi^i(\tau)$, where

$$|T_i\rangle = \frac{\partial g^a}{\partial \pi^i} |a\hbar\rangle = T_i^a |a\hbar\rangle. \quad (34)$$

Then the “acceleration” is as follows

$$|A\rangle = \frac{d^2|G\rangle}{d\tau^2} = |g_{ik}\rangle \frac{d\pi^i}{d\tau} \frac{d\pi^k}{d\tau} + |T_i\rangle \frac{d^2\pi^i}{d\tau^2} = |N_{ik}\rangle \frac{d\pi^i}{d\tau} \frac{d\pi^k}{d\tau} + \left(\frac{d^2\pi^s}{d\tau^2} + \Gamma_{ik}^s \frac{d\pi^i}{d\tau} \frac{d\pi^k}{d\tau}\right) |T_s\rangle, \quad (35)$$

where

$$|g_{ik}\rangle = \frac{\partial^2 g^a}{\partial \pi^i \partial \pi^k} |a\hbar\rangle = |N_{ik}\rangle + \Gamma_{ik}^s |T_s\rangle \quad (36)$$

and the state

$$|N\rangle = N^a |a\hbar\rangle = \left(\frac{\partial^2 g^a}{\partial \pi^i \partial \pi^k} - \Gamma_{ik}^s \frac{\partial g^a}{\partial \pi^s}\right) \frac{d\pi^i}{d\tau} \frac{d\pi^k}{d\tau} |a\hbar\rangle \quad (37)$$

is the normal to the “hypersurface” of the ground states. Then the minimization of this “acceleration” under the transition from point τ to $\tau+d\tau$ may be achieved by the annihilation of the tangential component

$$\left(\frac{d^2\pi^s}{d\tau^2} + \Gamma_{ik}^s \frac{d\pi^i}{d\tau} \frac{d\pi^k}{d\tau}\right) |T_s\rangle = 0 \quad (38)$$

i.e. under the condition of the affine parallel transport of the Hamiltonian vector field

$$dH^s + \Gamma_{ik}^s H^i d\pi^k = 0. \quad (39)$$

The derivatives arose in previous formulas are as following:

$$\begin{aligned} T_i^0 &= \frac{\partial g^0}{\partial \pi^i} = -\frac{1}{2} \frac{\pi^{*i}}{\left(\sqrt{\sum_{s=1}^{N-1} |\pi^s|^2 + 1}\right)^3}, \\ T_k^{0*} &= \frac{\partial g^0}{\partial \pi^{*k}} = -\frac{1}{2} \frac{\pi^k}{\left(\sqrt{\sum_{s=1}^{N-1} |\pi^s|^2 + 1}\right)^3} \end{aligned} \quad (40)$$

and for other components ($a \geq 1$) one has

$$\begin{aligned} T_i^m &= \frac{\partial g^m}{\partial \pi^i} = \left(\frac{\delta_i^m}{\sqrt{\sum_{s=1}^{N-1} |\pi^s|^2 + 1}} - \frac{1}{2} \frac{\pi^m \pi^{*i}}{\left(\sqrt{\sum_{s=1}^{N-1} |\pi^s|^2 + 1}\right)^3} \right), \\ T_k^{m*} &= \frac{\partial g^{*m}}{\partial \pi^{*k}} = \left(\frac{\delta_k^m}{\sqrt{\sum_{s=1}^{N-1} |\pi^s|^2 + 1}} - \frac{1}{2} \frac{\pi^{*m} \pi^k}{\left(\sqrt{\sum_{s=1}^{N-1} |\pi^s|^2 + 1}\right)^3} \right). \end{aligned} \quad (41)$$

At first sight they have neither connection with space-time (or momentum) representation which we need for a dynamical description. This representation one should obtain by the introduction of a local *dynamical space-time* [19, 20].

We saw that $SU(N)$ geometry gives the shape and the orientation of the ellipsoid associated with the “average” of dynamical variable given by a quadric form $< F | \hat{D} | F >$. If it is taking “as given” it show only primitive eigenvalue problem. But if one rises the question about real operational sense of the quantum measurement of this dynamical variable or the process of the transition from one eigen-state to another, one sees that quantum state and dynamical variable involved in much more complicated relations that it is given in the orthodox quantum scheme. The simple reason for this is that the decomposition (representation) of the state vector of a quantum system strongly depends on the spectrum and eigen-vectors of its dynamical variable. Overloaded system of the GCS’s supplies us by enough big “reserve” of functions but their superposition should be local and they span a tangent space at any specific point of $CP(N-1)$ marked by the local coordinates.

The “probability” may be introduced now by pure geometric way like $\cos^2 \phi$ in tangent state space as follows.

For any two tangent vectors $D_1^i = < D_1 | T_i >$, $D_2^i = < D_2 | T_i >$ one can define the scalar product

$$(D_1, D_2) = \Re G_{ik}^* D_1^i D_2^{k*} = \cos \phi_{1,2} (D_1, D_1)^{1/2} (D_2, D_2)^{1/2}. \quad (42)$$

Then the value

$$P_{1,2}(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1}) = \cos^2 \phi_{1,2} = \frac{(D_1, D_2)^2}{(D_1, D_1)(D_2, D_2)} \quad (43)$$

may be treated as a relative probability of the appearance of two states arising during the measurements of two different dynamical variables D_1, D_2 by the variation of the initial GCS $(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1})$.

Some LDV $\vec{\Psi} = \Psi^i \frac{\partial}{\partial \pi^i} + c.c.$ may be associated with the “state vector” $|\Psi\rangle \in \mathcal{H}$ which has tangent components $\Psi^i = \langle T_i | \Psi \rangle$ in $T_\pi CP(N-1)$. Thus the scalar product

$$(\Psi, D) = \Re G_{ik^*} \Psi^i D^{k^*} = \cos \phi_{\Psi, D} (\Psi, \Psi)^{1/2} (D, D)^{1/2}. \quad (44)$$

gives the local correlation between two LDV's at same GCS. The cosines of directions

$$P_{\Psi, i}(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1}) = \cos^2 \phi_{\Psi, i} = \frac{(\Psi, D^i)^2}{(\Psi, \Psi)(D^i, D^i)} \quad (45)$$

may be identified with “probabilities” in each tangent direction of $T_\pi CP(N-1)$. The conservation law of “probability” is given by the simple identity

$$\sum_{i=1}^{N-1} P_{\Psi, i} = \sum_{i=1}^{N-1} \cos^2 \phi_{\Psi, i} = 1. \quad (46)$$

The notion of the “probability” is of course justified by our experience since different kinds of fluctuations prevent the exact knowledge of any quantum dynamical variable. That is not only because the uncertainty relation between *two* canonically conjugated dynamical variables puts the limit of accuracy, but because any real measurement of a *single* dynamical variable or the process of preparation of some state are not absolutely exact. It is easy to see from the relation between the velocity $V^i = \frac{d\pi^i}{d\tau}$ in $CP(N-1)$ and the energy variance $(\Delta H)^2$ through Aharonov-Anandan relationship $\frac{dS}{d\tau} = \frac{2\Delta H}{\hbar}$ [21], where $\Delta H = \sqrt{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2}$ is the uncertainty of the Hamiltonian \hat{H} . Indeed, the quadric form in the local coordinates is as follows: $dS^2 = G_{ik^*} d\pi^i d\pi^{k^*} = \frac{4(\Delta H)^2}{\hbar^2} d\tau^2$ and, therefore,

$$(\Delta H)^2 = \frac{\hbar^2}{4} G_{ik^*} \frac{d\pi^i}{d\tau} \frac{d\pi^{k^*}}{d\tau}, \quad (47)$$

i.e. velocity V^i in $CP(N-1)$ defines the variance of the Hamiltonian.

But it is not the reason to deny a possibility to know any dynamical variable with an acceptable accuracy.

5 The geometric way of the gauge field generation

The fundamental gauge field coming “from nowhere” in the models of elementary particles, and both Abelian [22] and non-Abelian [23] pseudo-potentials associated with adiabatic Born-Oppenheimer approximation, have formally geometric origins but of a different nature. Pseudo-potentials have a singular source of monopole-like type whose nature arose under degeneration, etc. But the mathematical artefact (singularity of mapping) cannot be a reason for physical phenomenon. Dirac put the monopole as a physical source of electromagnetic field. He assumed that singularities are concentrated on some “line of the knots” in the physical space. However, the monopoles do not exist up to now as physical object; one should have clear mechanism of the non-integrability of the phase (or action functional).

The question is: is it possible to find a non-singular fundamental gauge potential if one uses not artificial parameter space, but an inherently related projective Hilbert space $CP(N-1)$.

I argue that in the framework of my model the reason of anholonomy is lurked in the curvature of the dynamical group manifold and its invariant submanifold $CP(N-1)$. The “fundamental” interaction is generated by the coset transformations [13]. Their geometry is the true source of some physical fields. It means that under the affine parallel transport of LDV’s agrees with the Fubini-Study metric and there is some anholonomy depending on the curvature of $CP(N-1)$.

If one uses the Berry formula (1.24) [11] in the local coordinates π^i

$$\begin{aligned} V_{ik*}(\pi^i) &= \Im \sum_{a=0}^{N-1} \left\{ \frac{\partial g^{a*}}{\partial \pi^i} \frac{\partial g^a}{\partial \pi^{k*}} - \frac{\partial g^{a*}}{\partial \pi^{k*}} \frac{\partial g^a}{\partial \pi^i} \right\} = \Im \sum_{a=0}^{N-1} \{ T_i^a T_k^{a*} - (T_k^a)^* T_i^a \} \\ &= -\Im [(1 + \sum |\pi^s|^2) \delta_{ik} - \pi^{i*} \pi^k] (1 + \sum |\pi^s|^2)^{-2} = -\Im G_{ik*}, \quad (48) \end{aligned}$$

one will find that it is closely related to Fubini-Study metric (quantum metric tensor). There are two important differences between original Berry’s formula referring to arbitrary parameters and this 2-form in local coordinates inherently connected with eigen-problem.

1. The $V_{ik*}(\pi^i)$ is the singular-free expression.
2. It does not contain two eigen-values, say, E_n, E_m explicitly, but implicitly V_{ik*} depends locally on the choice of single λ_p through the dependence in local coordinates $\pi_{j(p)}^i$.

6 Objective Quantum Measurement

New scientific paradigm concerns the concept of the Multiverse or omnium [1] where universal laws of Universe may be contravened. I assume that this assumption is premature since it is based on very speculative models of extended

quantum objects. I assume instead that our Universe has in fact infinite dimension (even the state of a single hydrogen atom belongs to Hilbert space) but quantum measurement reduce locally the Universe Geometry (UG) down to 4D dynamical space-time whose geometry is a function of the state of the measurement setup. It means that metric, connection, curvature, etc. of the dynamical space-time (DST), are microscopically state-dependent. Such reduction during objective quantum measurement will be described below; this procedure leads to different kind of the extended quantum objects - “field shell” of GCS.

Definitely, it is impossible mathematically describe the state of some real quantum setup. I will use a GCS $(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1})$ of some action operator $\hat{S} = \hbar A(\hat{\eta}^\dagger \hat{\eta})$ representing physically distinguishable states. This means that any two points of $CP(N-1)$ define two ellipsoids differ at least by the orientations, if not by the shape, as it was discussed above.

I assume that there is *expectation state* $|D\rangle: \hat{D}|D\rangle = \lambda_p|D\rangle$, associated with “measuring device” tuned for measurement of dynamical variable \hat{D} at some eigen-state $(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1})$

$$|D\rangle = |D_p(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1})\rangle = \sum_{a=0}^{N-1} g^a(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1}) |\hbar a\rangle = \sum_{a=0}^{N-1} g^a |\hbar a\rangle. \quad (49)$$

Hereafter I will omit indexes $j(p)$ for a simplicity. Now one should build the spinor of the “logical spin 1/2” in the local basis $(|N\rangle, |\tilde{D}\rangle)$ for the quantum question in respect with the measurement of the local dynamical variable \tilde{D} at corresponding GCS which may be marked by the local normal state

$$|N\rangle = N^a |\hbar a\rangle = \left(\frac{\partial^2 g^a}{\partial \pi^i \partial \pi^k} - \Gamma_{ik}^s \frac{\partial g^a}{\partial \pi^s} \right) \frac{d\pi^i}{d\tau} \frac{d\pi^k}{d\tau} |\hbar a\rangle. \quad (50)$$

Since in general $|D\rangle$ it is not a tangent vector to $CP(N-1)$, the deviation from GCS during the measurement of \tilde{D} will be represented by tangent vector

$$|\tilde{D}\rangle = |D\rangle - \langle Norm|D\rangle |Norm\rangle = |D\rangle - \langle N|D\rangle \frac{|N\rangle}{\langle N|N\rangle} \quad (51)$$

defined as the covariant derivative on $CP(N-1)$. This operation is the orthogonal projector \hat{Q} . Indeed,

$$\begin{aligned} \widetilde{|\tilde{D}\rangle} &= (|D\rangle - \langle Norm|D\rangle |Norm\rangle) \\ &= |D\rangle - \langle Norm|D\rangle |Norm\rangle \\ &- \langle Norm|(|D\rangle - \langle Norm|D\rangle |Norm\rangle) |Norm\rangle \\ &= |D\rangle - \langle Norm|D\rangle |Norm\rangle = |\tilde{D}\rangle. \end{aligned} \quad (52)$$

This projector \hat{Q} takes the place of dichotomic dynamical variable (quantum question) for the discrimination of the normal state $|N\rangle$ (it represents the eigen-state at GCS) the and the orthogonal tangent state $|\tilde{D}\rangle$ that represents

the velocity of deviation form GCS. The coherent superposition of two eigenvectors of \hat{Q} at the point $(\pi^1, \dots, \pi^{N-1})$ forms the spinor η with the components

$$\begin{aligned}\alpha_{(\pi^1, \dots, \pi^{N-1})} &= \frac{\langle N|D \rangle}{\langle N|N \rangle} \\ \beta_{(\pi^1, \dots, \pi^{N-1})} &= \frac{\langle \tilde{D}|D \rangle}{\langle \tilde{D}|\tilde{D} \rangle}.\end{aligned}\quad (53)$$

Then from the infinitesimally close GCS $(\pi^1 + \delta^1, \dots, \pi^{N-1} + \delta^{N-1})$, whose shift is induced by the interaction used for a measurement, one get a close spinor $\eta + \delta\eta$ with the components

$$\begin{aligned}\alpha_{(\pi^1 + \delta^1, \dots, \pi^{N-1} + \delta^{N-1})} &= \frac{\langle N'|D \rangle}{\langle N'|N' \rangle} \\ \beta_{(\pi^1 + \delta^1, \dots, \pi^{N-1} + \delta^{N-1})} &= \frac{\langle \tilde{D}'|D \rangle}{\langle \tilde{D}'|\tilde{D}' \rangle},\end{aligned}\quad (54)$$

where the basis $(|N' \rangle, |\tilde{D}' \rangle)$ is the lift of the parallel transported $(|N \rangle, |\tilde{D} \rangle)$ from the infinitesimally close point $(\pi^1 + \delta^1, \dots, \pi^{N-1} + \delta^{N-1})$ back to the $(\pi^1, \dots, \pi^{N-1})$. It is clear that such parallel transport should be somehow connected with the variation of coefficients Ω^α in the dynamical space-time.

The covariance relative transition from one GCS to another

$$(\pi_{j(p)}^1, \dots, \pi_{j(p)}^{N-1}) \rightarrow (\pi_{j'(q)}^1, \dots, \pi_{j'(q)}^{N-1}) \quad (55)$$

and the covariant differentiation (relative Fubini-Study metric) of vector fields provides the objective character of the “quantum question” \hat{Q} and, hence, the quantum measurement. This serves as a base for the construction of the dynamical space-time as it will be shown below.

These two infinitesimally close spinors may be expressed as functions of θ, ϕ, ψ, R and $\theta + \epsilon_1, \phi + \epsilon_2, \psi + \epsilon_3, R + \epsilon_4$, and represented as follows

$$\eta = R \begin{pmatrix} \cos \frac{\theta}{2} (\cos \frac{\phi - \psi}{2} - i \sin \frac{\phi - \psi}{2}) \\ \sin \frac{\theta}{2} (\cos \frac{\phi + \psi}{2} + i \sin \frac{\phi + \psi}{2}) \end{pmatrix} = R \begin{pmatrix} C(c - is) \\ S(c_1 + is_1) \end{pmatrix} \quad (56)$$

and

$$\begin{aligned}\eta + \delta\eta &= R \begin{pmatrix} C(c - is) \\ S(c_1 + is_1) \end{pmatrix} \\ + R \begin{pmatrix} S(is - c)\epsilon_1 - C(s + ic)\epsilon_2 + C(s + ic)\epsilon_3 + C(c - is)\frac{\epsilon_4}{R} \\ C(c_1 + is_1)\epsilon_1 + S(ic_1 - s_1)\epsilon_2 - S(s_1 - ic_1)\epsilon_3 + S(c_1 + is_1)\frac{\epsilon_4}{R} \end{pmatrix}\end{aligned}\quad (57)$$

may be connected with infinitesimal “Lorentz spin transformations matrix” [24]

$$L = \begin{pmatrix} 1 - \frac{i}{2}\tau(\omega_3 + ia_3) & -\frac{i}{2}\tau(\omega_1 + ia_1 - i(\omega_2 + ia_2)) \\ -\frac{i}{2}\tau(\omega_1 + ia_1 + i(\omega_2 + ia_2)) & 1 - \frac{i}{2}\tau(-\omega_3 - ia_3) \end{pmatrix}. \quad (58)$$

Then accelerations a_1, a_2, a_3 and angle velocities $\omega_1, \omega_2, \omega_3$ may be found in the linear approximation from the equation

$$\eta + \delta\eta = L\eta \quad (59)$$

as functions of the “logical spin $1/2$ ” spinor components depending on local coordinates $(\pi^1, \dots, \pi^{N-1})$.

Hence the infinitesimal Lorentz transformations define small “space-time” coordinates variations. It is convenient to take Lorentz transformations in the following form $ct' = ct + (\vec{x}\vec{a})d\tau$, $\vec{x}' = \vec{x} + ct\vec{a}d\tau + (\vec{\omega} \times \vec{x})d\tau$, where I put $\vec{a} = (a_1/c, a_2/c, a_3/c)$, $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ [24] in order to have for τ the physical dimension of time. The coordinates x^μ of points in this space-time serve in fact merely for the parametrization of deformations of the “field shell” arising under its motion according to non-linear field equations [19, 20, 25].

7 Field equations in the dynamical space-time

The energetic packet - “particle” associated with the “field shell” is now described locally by the Hamiltonian vector field $\vec{H} = \hbar\Omega^\alpha\Phi_\alpha^i\frac{\partial}{\partial\pi^i} + c.c.$ Our aim is to find the wave equations for Ω^α in the dynamical space-time intrinsically connected with the objective quantum measurement.

At each point $(\pi^1, \dots, \pi^{N-1})$ of the $CP(N-1)$ one has an “expectation value” of the \vec{H} defined by a measuring device. But a displaced GCS may be reached along one of the continuum paths. Therefore the comparison of two vector fields and their “expectation values” at neighboring points requires some natural rule. The comparison makes sense only for the same “particle” or for its “field shell” along some path. For this reason one should have an identification procedure. The affine parallel transport in $CP(N-1)$ of vector fields is a natural and the simplest rule for the comparison of corresponding “field shells”. Physically the identification of “particle” literally means that its Hamiltonian vector field is a Fubini-Study covariant constant.

Let me apply this parallel transport to the local Hamiltonian vector field $H^i = \hbar\Omega^\alpha\Phi_\alpha^i$. The invariant classification of the LDV’s given above shows that $\vec{H}_b = \hbar\Omega^b\Phi_b^i\frac{\partial}{\partial\pi^i}$ describes the velocity of the shape variation of the “ellipsoid of polarization”, but $\vec{H}_h = \hbar\Omega^h\Phi_h^i\frac{\partial}{\partial\pi^i}$ gives the variation velocity of the orientation of this ellipsoid in iso-space. Therefore Ω^h may be treated as “pure gauge fields” whereas Ω^b are “matter field” whose the mass spectrum $M_b = \hbar\Omega^b/c^2$ gives the axes of the deformed ellipsoid.

Since we have only the unitary fields Ω^α as parameters of the GCS transformations we assume that in accordance with an super-equivalence principle [4, 13] under the infinitesimal variation of the unitary field $\delta\Omega^\alpha$ in the dynamical space-time, the shifted Hamiltonian field should coincide with the infinitesimal shift of the tangent Hamiltonian field generated by the parallel transport in $CP(N-1)$ during world time $\delta\tau$ [18]. Thus one has

$$\hbar(\Omega^\alpha + \delta\Omega^\alpha)\Phi_\alpha^k = \hbar\Omega^\alpha(\Phi_\alpha^k - \Gamma_{mn}^k\Phi_\alpha^m V^n\delta\tau) \quad (60)$$

and, hence, in accordance with the sufficiency criterion for the equation with non-trivial solution

$$\hbar(\delta\Omega^\alpha\delta_m^k + \Omega^\alpha\Gamma_{mn}^k V^n\delta\tau)\Phi_\alpha^m = 0 \quad (61)$$

one has following equations

$$\frac{\delta\Omega^\alpha}{\delta\tau} = -\Omega^\alpha\Gamma_{mn}^m V^n, \quad (62)$$

(there is no summation in m).

We introduce the dynamical space-time coordinates x^μ as state-dependent quantities, transforming in accordance with the local Lorentz transformations $x^\mu + \delta x^\mu = (\delta_\nu^\mu + \Lambda_\nu^\mu\delta\tau)x^\nu$. The parameters of $\Lambda_\nu^\mu(\pi^1, \dots, \pi^{N-1})$ depend on the local transformations of local reference frame in $CP(N-1)$ described in the previous paragraph. Then taking into account the expressions for the “4-velocity” $v^\mu = \frac{\delta x^\mu}{\delta\tau} = \Lambda_\nu^\mu(\pi^1, \dots, \pi^{N-1})x^\nu$ where

$$\begin{aligned} v^0 &= (\vec{x}\vec{a}) \\ \vec{v} &= ct\vec{a} + (\vec{\omega} \times \vec{x}), \end{aligned} \quad (63)$$

one has the field equations

$$v^\mu \frac{\partial\Omega^\alpha}{\partial x^\mu} = -\Omega^\alpha\Gamma_{mn}^m V^n. \quad (64)$$

One has in fact $N-1$ equations which may refer to the spectrum generating fields Ω^b . There are some questions concerning this construction. First: the Dirac matrices $\gamma_{ss'}^\mu$ have the sense of some velocities, but the possible connection between them and v^μ is not clear. The second question is: how one can get $(N-1)^2$ equations for pure gauge fields Ω^h ? Probably they relate to Killing fields in $CP(N-1)$, but I have not now answers on these questions.

CONCLUSION

1. It is proposed the generalized (in comparison with “2-level” case [4]) the intrinsically geometric scheme of the quantum measurement of an arbitrary Hermitian “N-level” dynamical variable. The interaction arose due to the breakdown of $G = SU(N)$ symmetry is used for such measurement and it is represented by the affine gauge “field shell” propagated in the dynamical state-dependent space-time.

2. The concept of “super-relativity” [13] is in fact a different kind of attempts of “hybridization” of the internal and space-time symmetries. In distinguish from SUSY where a priori exists the “extended space-time - super-space”, in my approach the dynamical space-time arises only as a parametrization of $SU(N)$ local dynamical variables under “yes/no” quantum measurement.

3. The pure local formulation of theory in $CP(N-1)$ leads seemingly to the decoherence [4]. We may, of course, to make mentally the concatenation of any two quantum systems living in direct product of their state spaces. The variation of the one of them during a measurement may lead formally to some

variations in the second one. Unavoidable fluctuations in our devices may even confirm predictable correlations. But the introduction of the state-dependent dynamical space-time evokes a necessity to reformulate the Bell's inequalities which may lead then to a different condition for the coincidences.

4. The locality in the quantum phase space $CP(N-1)$ leads to extended quantum particles - "field shell" that obey the quasi-linear PDE [19, 20, 25]. The physical status of their solutions is the open question. But if they somehow really connected with "elementary particles", say, electrons, then the plane waves of de Broglie should not be literally refer to the state vector of the electron itself but rather to covector (1-form) realized, say, as a periodic cristall lattice. The fact that the condition for diffraction is in nice agreement with experiments may be explained that for this agreement it is important only *relative velocity* of electron and the lattice. The fact that momentum of electron $p_k = \frac{\partial W}{\partial q^k}$ and the group velocity $v^k = \frac{dq^k}{dt}$ should be in opposite directions may be easy proved. Schrödinger [9] pointed out that the Hamilton-Jacobi equation for the material point with the unit mass $m = 1$

$$\frac{\partial W}{\partial t} + T(q^k, \frac{\partial W}{\partial q^k}) + V(q^k) = 0 \quad (65)$$

may be understood as describing the surfaces of constant phase of the action waves. If

$$W = -Et + S(q^k), \quad (66)$$

then one gets

$$T(q^k, \frac{\partial W}{\partial q^k}) = E - V(q^k). \quad (67)$$

The Hertz metric

$$dS^2 = 2T(q^k, \frac{dq^k}{dt})dt^2 = 2(E - V(q^k))dt^2 = g_{ik}dq^i dq^k. \quad (68)$$

may be interpreted as a some effective "index of refraction"

$$n = \sqrt{2T} = \sqrt{g_{ik}v^i v^k} = \sqrt{2(\hbar\omega - V)} \quad (69)$$

for the action waves with the positive phase velocity

$$u = \frac{E}{\sqrt{2T}} = \frac{E}{\sqrt{g_{ik}v^i v^k}} = \frac{\hbar\omega}{\sqrt{2(\hbar\omega - V)}}. \quad (70)$$

But the scalar product

$$p_k v^k = \frac{\partial W}{\partial q^k} \frac{dq^k}{dt} = \frac{\partial W}{\partial t} = -E \quad (71)$$

shows that the momentum and the group velocity should be in opposite directions.

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